reveal.) It would be appropriate to name your child after either of them. But since I want to give you some options, we can also consider Vardisha and Vardiqua." To Cousin Noah's great surprise, Christin agrees to allow Mr. G. to choose one of these four names – but only if he can solve a related math problem.

"Listen carefully, Mr. G," begins Christin. "The four names – Vardetta, Vardiga, Vardisha, and Vardiqua – are arranged in ascending order by total value, calculated by the sum of all letters. Each vowel is worth 10, 20, 30, or 40, while each consonant is worth a prime number less than 20. Different letters represent different values. For example, if D = 2, then no other letter can be 2.

"Furthermore, an A in one name is worth the same as an A in another name. So if A = 20 in Vardetta, then A = 20 in Vardiga, Vardisha, and Vardiqua. This is true for all letters: the value of a letter in one name is the same as the value of that identical letter in any other name.

"The total value of the name Vardetta is 100 (thus, V+A+R+D+E+T+T+A = 100). The total value of the name Vardiga is 3 more than the total value of Vardetta, while the total value of Vardiqua is S more than the total value of Vardisha. If A > E, what is the 2^{nd} highest possible value of VARDISHA + VARDIQUA?"

SOLUTIONS: Nathaniel Satriya Division

1) **½ (or 50% or .5)** **Logic or Probability/CLT.** *Using logic is much faster, but not everyone will recognize that method when solving this problem.*

<u>Method 1: Logic</u>. Kavya has one more coin than Kriti. If Kavya does not throw more heads than Kriti, then she <u>must</u> throw more tails than Kriti. (Note that the number of heads could be the same, but in that case, Kavya would throw one more tails than Kriti since she has an extra quarter.)

This means that the probability of Kavya throwing more heads plus the probability of Kavya throwing more tails equals 1 (or 100%).

The probability of throwing heads and tails is equal, so the probability that Kavya throws heads must be ½.

<u>Method 2: Probability CLT</u>. Most questions like this need to be solved the long way by calculating all the cases and making a chart. First, start with Kavya. The probability of her throwing 0 heads is ½ x ½ x ½ x ½ x ½ x ½ = 1/64. The same is true of the probability of her throwing 6 heads.

For 1 heads, Kavya can get heads first, second, third, fourth, fifth, or sixth. That's 6 ways out of 64. The same is true for 5 heads.

For 2 heads, you need to list the ways in order. By rows, we get:

HH----	H-H---	H--H--	H---H-	H----H
-HH---	-H-H--	-H--H-	-H---H	
--HH--	--H-H-	--H--H		
---HH-	---H-H			
----HH				

That is 15 ways out of 64. Note that this amount can also be calculated using nCr (combinations): in this case, it's 6 choose 2, which is 6! / 2! (6-2)! = 6! / 2! 4! = (6 x 5) / (2 x 1) = 30 / 2 = 15.

What is true for 2 heads will also work for 4 heads. Thus, we have 1 + 1 + 6 + 6 + 15 + 15 = 44 ways so far. The only thing left is for 3 heads, but instead of listing, we can subtract. 64 – 44 = 20 ways left. Once again, this amount can also be calculated using nCr (combinations): 6 choose 3 is 6! / 3! (6-3)! = 6! / 3! 3! = (6 x 5 x 4) / (3 x 2 x 1) = 120 / 6 = 20.

This chart summarizes the results.

# of Heads	# of Ways out of 64
0	1
1	6
2	15
3	20
4	15
5	6
6	1

Do the same thing for Kriti. To get 0 heads, her probability is ½ x ½ x ½ x ½ x ½ = 1/32. The same is true for her to get 5 heads.

To get 1 heads, she can get heads first, second, third, fourth, or fifth. That's 5 ways out of 32. The same is true for 4 heads.

For 2 heads, list the ways (or use nCr):

HH---	H-H---	H--H-	H---H
-HH--	-H-H-	-H--H	

--HH- --H-H
---HH

That's 10 ways to make 2 heads. The same is true for 3 heads. Kriti's chart is shown below.

# of Heads	# of Ways out of 32	# of Ways out of 64
0	*1*	*2*
1	*5*	*10*
2	*10*	*20*
3	*10*	*20*
4	*5*	*10*
5	*1*	*2*

Kriti's results have been converted to # of Ways out of 64 to be consistent with Kavya's.

To calculate the probability that Kavya will throw more heads than Kriti, we have to consider each case. If Kavya throws 0 heads, her chance of winning is 0 (since she there is no way Kriti can throw a negative number of heads). If Kavya throws 1 heads, her chance of winning is 2/64, because the only way she can win is if Kriti throws 0 heads (see chart above).

Similarly, if Kavya throws 2 heads, her chance of winning is 2 + 10 out of 64, or 12/64 (this means that Kriti can throw either 0 heads or 1 heads). Use the same logic to determine Kavya's chance of winning with 3 heads (32/64), 4 heads (52/64), 5 heads (62/64), and 6 heads (100%, or 64/64). The results can be seen in the chart below.

# of Heads for Kavya	Probability of Kavya throwing this # of Heads)	Kavya's Win Probability vs. Kriti
0	*1/64*	*0/64*
1	*6/64*	*2/64*
2	*15/64*	*12/64*
3	*20/64*	*32/64*
4	*15/64*	*52/64*
5	*6/64*	*62/64*
6	*1/64*	*64/64*

To find the total overall probability that Kavya will win, we must combine all these probabilities together. This seems like a lot of work, but since the denominators are all 64, we can expect it to work out reasonably well. Don't do the complicated multiplications until you have to. LOOK at the numbers and USE DISTRIBUTIVE PROPERTY! I have paired up similar fractions using different fonts.

(1/64 x 0/64) + (6/64 x 2/64) + **(15/64 x 12/64)** + (20/64 x 32/64) +
(15/64 x 52/64) + (6/64 x 62/64) + (1/64 x 64/64) =

0 + (6/64 x 1) + **(15/64 x 1)** + (20/64 x 1/2) + (1/64 x 1) =
0 + 6/64 + 15/64 + 10/64 + 1/64 =
(0 + 6 + 15 + 10 + 1) / 64 =
32/64 = ½.

2) **40** **Logic; Process of Annihilation.** *Originally, I solved this problem by focusing on the middle and right columns, listing all possible digits that could go in those boxes. I determined that the only two digits that didn't work for any of those boxes were 5 and 7; therefore, those digits had to go in the left column. I then figured out which numbers went in which box from there.*

That method works, and it did lead to the correct answer. However, there is a more direct method which is faster and easier. ***Nathaniel*** *provides it as follows:*

This problem has a lot of different restrictions. To make it easier from the start, <u>let's begin solving by using the most restrictive fact: that $C \times V \div Z = 27$</u>. We can see from this that $C \times V$ has to be a multiple of 27, so C and V have to be either 9 and 3 or 9 and 6.

If they are 9 and 3, Z must be 1. However, since $X \div Y - Z = 0$, there is a problem: if Z is 1, then $X \div Y$ must also be 1, which means that $X = Y$. That is not possible, so ***Z must be 2****, and C and V are 9 and 6.*

Now since $T \div U - V = 1$, V must be 6: if it were 9, then $T \div U = 10$, which is again impossible. Therefore, ***V = 6, meaning that C = 9*** *and $T \div U = 7$. It is then clear that* ***T is 7 and U is 1****, as those are the only digits whose quotient is 7.*

Now, notice that $X \div Y - Z = 0$. Since $Z = 2$, $X \div Y$ must also be 2. Since 2 and 6 have already been used, ***X and Y have to be 8 and 4 respectively****.*

Finally, because $B \div U \times Y = 12$, ***B is 3*** *($3 \times 4 = 12$)* ***and A, the last remaining digit, is 5*** *($5 + 3 + 9 = 17$). Our final answer, $Y \times A \times Z$, is $4 \times 5 \times 2$, or* ***<u>40</u>****.*

The completed chart looks like this:

5	**3**	**9**
7	**1**	**6**
8	**4**	**2**

3) **– 6** **Cancelling/Reducing.** *On several contests this year, when the TA's looked at solutions to verify that work had been shown, they saw huge cross outs until the only thing left for each numerator and denominator were whole numbers. In effect, that's all this problem is: reducing everything until it's as simple as possible. We call this problem type a Nathaniel Special.*

Because the first fraction is not being multiplied by the second or third, it must be reduced independently. We will start there.

<u>$(2^{16} \times 3^{27} \times 5^5) / (30^5 \times 18^{11})$</u>: $2 \times 3 \times 5 = 30$, so $2^5 \times 3^5 \times 5^5 = 30^5$. We are left with $(2^{11} \times 3^{22}) / 18^{11}$. $2 \times 3 \times 3 = 18$, so $2^{11} \times 3^{11} \times 3^{11} = 18^{11}$. The numerator and denominator

*completely cancel out, leaving us with **1** for the first fraction.*

The second and third fractions are multiplied to each other, so we can cancel numerators with denominators from either fraction. Starting with the third fraction: take out 77^2 immediately. This leaves us with 7 / (11 x 77^2). Now cancel out a 7. The third fraction becomes 1 / (11 x 11 x 77). Rewrite this as 1 / (11 x 11 x 11 x 7), which is 1 / (11^3 x 7).

Multiply this to the second fraction to get (7^5 x 11^4 x 13^3) / (1001 x 91^2 x 11^3 x 7). Cancel out the obvious things first: one 7 and 11^3. This leaves us with (7^4 x 11 x 13^3) / (1001 x 91^2). Notice that 7 x 13 = 91. Then 7^2 x 13^2 = 91^2. Cancel again to get (7^2 x 11 x 13) / 1001. At this point, your instincts should tell you that 1001 probably equals 7 x 11 x 13. Indeed, 7 x 11 = 77, and 77 x 13 = 1001. Cancel one more time, and you are left with 7 x 1 x 1 / 1 = 7.

The first fraction reduces to 1; the second and third fractions reduce to 7. Then 1 – 7 = **–6**.

4) **387/512** **Probability.** *For this type of problem, consider each result separately and STAY ORGANIZED because there is much to do!*

<u>Case 1: Roll of 6-sided die is a prime number</u>. On a 6-sided die, the prime numbers are 2, 3, and 5, so the chance of rolling a prime number is 3/6 = ½. At this point, you flip a coin.

- *Heads = ½ = Loss.*
- *Tails = ½ = Roll 12-sided die. Even number = win; chance of even # is ½, so chance of winning is ½ x ½ = ¼. Then chance of losing is also ¼.*
- *Combine these results: ¼ chance of winning and (½ + ¼) = ¾ chance of losing.*

<u>Case 2: Roll of 6-sided die is not a prime number</u>. 1, 4, and 6 are not prime, so the chance of rolling a non-prime number is 3/6 = ½. Now, you roll the 12-sided die.

- *Odd # = win. Chance of odd is ½.*
- *Even # = loss. Chance of even is also ½.*
- *This means that there is a ½ chance of winning and a ½ change of losing.*

<u>Multiply each case by ½</u>. Since Case 1 and Case 2 have the same probability of occurring, the total probability on each turn of a win and loss can be calculated as follows:

Case 1 Win: ½ x ¼ = 1/8. *Case 1 Loss: ½ x ¾ = 3/8.*
Case 2 Win: ½ x ½ = ¼. *Case 2 Loss: ½ x ½ = ¼.*

<u>Win probability per turn</u> = 1/8 + ¼ = 3/8.
<u>Loss probability per turn</u> = 3/8 + ¼ = 5/8.

Rather than determining the probability of winning at least once in the first three tries, it is much easier to calculate the probability of NOT winning all three times, since that is the only result that doesn't give us what we want. The probability of NOT winning on one try is 5/8, so the probability of not winning on three consecutive tries is 5/8 x 5/8 x 5/8 = 125/512.

*The total probability of all outcomes is 512/512, so the probability of winning at least once in the first three tries is 512/512 – 125/512 = **387/512**.*

5) **234,567** **Smaller Case; Logic.** *With dice that have 739,940 sides, it is clearly not feasible to list all the possibilities using a chart! Instead, use a smaller case to illustrate what will happen. Let us*

say that Bidipta has two 6-sided dice, the first of which is painted blue on all faces except for one. To determine how many faces the second die must be painted gold in order for Bidipta and Pratham to have an equal chance of winning, first use logic, then use a chart.

The first die has 5B and 1G face. If the second die has:

- *0G faces, it will have 6B faces and Bidipta will win almost all the time (5 x 6 = 30 ways out of 36.)*
- *1G face, it will have 5B faces and Bidipta will win most of the time (more than 5 x 5 = 25 out of 36).*
- *2G faces, it will have 4B faces and Bidipta still wins most of the time (more than 5 x 4 = 20 out of 36).*
- *3G faces, then it will have 3B faces, and Bidipta will win more than 5 x 3 = 15 out of 36. This is the first time Bidipta has less than a ½ chance of winning thus far, so it is time to make a chart.*

	Blue	**Blue**	**Blue**	*Gold*	*Gold*	*Gold*
Blue	*B*	*B*	*B*			
Blue	*B*	*B*	*B*			
Blue	*B*	*B*	*B*			
Blue	*B*	*B*	*B*			
Blue	*B*	*B*	*B*			
Gold				*B*	*B*	*B*

We can see from this chart that Bidipta will win 18 out of 36 times, which is exactly ½. This result occurs when 3 out of 6 sides (or ½ of the sides) have been painted gold. We conclude that Bidipta will need to paint ½ of the sides of the second die gold in order to give himself and Pratham an equal chance of winning. ½ of 739,940 = ***369,970****, which is the value of G.*

Now that we know the value of G, use logic to determine S, A, and R, and finally calculate B. To make B as small as possible, we need to make S, A, and R as large as possible – specifically, we need $S^4 + A^3 + R^2$ *to be as large as possible. Since S, A, and R are distinct prime numbers less than 20, assign the highest value to S (since it has the largest power), the second highest to A, and the third largest to R. This means that S = 19, A = 17, and R = 13. We get*

$$19^4 + 17^3 + 13^2 + B = 369{,}970$$
$$130{,}321 + 4913 + 169 + B = 369{,}970$$
$$135{,}403 + B = 369{,}970$$
$$B = \mathbf{234{,}567}$$

6) **29 5/11** **Method 1: Nathaniel's Method: a fun combination of multiple different techniques.**

To start the problem, let's use what is given by finding the obtuse angle formed by the hands of the clock at 12:30. At 12:30, the minute hand is halfway around the clock, at the 180 degree position. The hour hand is 1/2 of the way between the large 12 and 1 of the clock, which respectively are at the 0 and 30 degree positions. Thus, the hour hand is at the (0+30) / 2 = 15 degree position, which means that the obtuse angle formed by the hands has a measure of 180 – 15 = 165 degrees.

The problem asks us how long it will be before an angle 1/5 of 165 = 33 degrees is first formed. Observe that the angle formed by the hands at 1:00 is exactly 30 degrees, which is quite close to what we want.

Also notice that for each minute, the minute hand moves 360/60 = 6 degrees forward while the hour hand moves 30/60 = 1/2 a degree forward. Subsequently (because the minute hand is behind the hour hand at this time), we know that ***the angle between the hands gets smaller at a rate of 6 – 1/2 = 11/2 degrees per minute.***

The observation that the angle between the hands changes at a linear rate is crucial to solving this problem. Since the angle measure decreases by 11/2 degrees per minute, it follows that the angle measure decreases by a degree every 2/11 minutes. At 1:00, or 30 minutes after 12:30, the angle between the hands measures 30 degrees. To get an angle of 33 degrees, we must add 3 degrees, which is equivalent to subtracting 3 x (2/11) = 6/11 minutes from the elapsed time. Thus, our final answer is 30 – 6/11 = ***<u>29 5/11 minutes</u>****.*

<u>Method 2: The Mr. G. Way: Linear Equations</u>. *Solve as above to find that the obtuse angle at 12:30 is 165 degrees and the desired angle is 165 / 5 = 33 degrees. Consider that the angle formed as the clock moves from 12:30 towards 1:00 decreases at a <u>linear rate</u>. This means that the angle gets smaller at a consistent, even rate: there are no jumps or curves along this path. The rate at which this angle gets smaller should correspond to a linear equation's slope.*

To determine this slope, consider two points along a specific line. At 12:30, the clock forms an obtuse angle of 165 degrees, but the angle between the hands as the clock approaches 1:00 is actually a reflex angle of 360 – 165 = 195 degrees. So at the current time, the angle between the hands is 195. Set x = 0 for the number of minutes after 12:30 and y = 195 for the degree measure between the hands.

Similarly, we know that when the clock moves forward to exactly 1:00, the angle between the hands will be equal to exactly 1/12 of a circle, which is 360 / 12 = 30 degrees. This means that at x = 30 (30 minutes after 12:30), y = 30 (the degree measure between the hands).

We now have two points for our linear equation: (0,195) and (30,30). The slope of the line connecting these points is then $\Delta y / \Delta x = (195 - 30) / (0 - 30) = 165 / -30 = -165 / 30$.

We are looking for the number of minutes that makes the angle 33 degrees. In other words, we seek the value of x that allows y to be 33. Using slope-intercept form, we can create a linear equation using y = mx + b, with m = –165 / 30 and b = 195 (recall that b is the y-intercept of the line, which is the value of y where x = 0). Our equation looks like this:

$$y = mx + b$$
$$33 = (-165 / 30)\, x + 195$$
$$-162 = (-165 / 30)\, x$$
$$x = \frac{-162\,(30)}{-165} = \frac{162\,(30)}{165} = \frac{162\,(2)}{11}$$
$$x = \mathbf{29\ 5/11}$$

7) **574 Logic (Cryptarithm); Process of Annihilation**. *This is a cryptarithm, but it's a hard one. As usual, start out by creating your Box O' Digits and make sure you understand the information before you begin solving.*

We seek the highest possible value of HOW when THAT has been maximized. This means that we must first find the greatest possible value of THAT.

<u>Begin with the thousands digit T</u>. If the three-digit numbers all began with 9, it would be possible for T to be as high as 3 due to carryovers from the 10's places. But since H, D, and Y must be different, the highest possible values for them would be 9, 8, and 7 (in some order). That would make 900 +

800 + 700 = 2400 for the hundreds places. For the tens and ones places, the maximum value will be less than 100. This means that OW + ID + OU + DO must be less than 4 x 100 = 400. Then the maximum value for THAT must be less than 2400 + 400 = 2800.

This means that T can be no greater than 2, while H must be less than 8. Since we must make THAT as large as we can, try <u>T = 2 and H = 7 first</u>.

If H = 7, then the greatest values we can have in the 100's places (for H, D, and Y) are 7, 9, and 8 (or 7, 8, and 9). We get 700 + 900 + 800 = 2400. We would then need a carryover of 3 from the 10's places to allow THAT to reach 2700. D could be 9 (to correspond with the D in the 100's place), but O and I cannot be 7, 8, or 9 because those digits have already been used. The most we could get in the 10's places would be D = 9, O = 6, and I = 5. Then OW + ID + OU + DO will be less than 70 + 60 + 70 + 100 = 300. Since the carryover from a sum less than 300 will be less than 3, this does not work.

Furthermore, if H = 7 and D and Y are the second highest possible combination (9 and 6), we would get 700 + 900 + 600 = 2200. There is no way we can get a carryover of 5 from the 10's place to make THAT at least 2700. ***This proves that H cannot be*** *7.*

<u>Now try H = 6</u>. Using a similar strategy, set D = 9 and Y = 8 to get the maximum possible value of THAT. 600 + 900 + 800 = 2300, meaning we would need a carryover of 3 from the 10's places. Reuse D = 9 for DO. Then the highest O can be is 7, and the highest I could be is 5. Then OW + ID + OU + DO will be less than 80 + 60 + 80 + 100 = 320. This ***could*** *work, so proceed.*

```
   6 7 W
   9 5 9
   8 7 U
+    9 7
--------
 2 6 A 2
```

The remaining digits for W, U, and A are 0, 1, 3, 4. In the 1's place, W + U + 16 must end with 2; the only possible sum is 22, which means W + U must be 6. There is no combination of remaining digits that adds to 6.

Instead, if we try I = 4 in place of I = 5, we get

```
   6 7 W
   9 4 9
   8 7 U
+    9 7
--------
 2 6 A 2
```

In the 10's place, 7 + 4 + 7 + 9 = 27. We need to make that number at least 30 so that there is a carry of 3 for the 100's place. To do this, we need to get a carryover of 3 from the 1's place (30 – 27 = 3). That means that W + U + 16 = 32, so W + U = 16. Since 7, 8, and 9 have already been used, there is no way to make this happen.

Finally, if we lower the value of O, then OW + ID + OU + DO will be less than 300, which will not give us a carryover of 3. This means that using 6, 9, and 8 for H, D, and Y will not work.

Since 6, 9, and 8 do not work for H, D, and Y, we now try the second highest possible combination: H = 6, D = 9, and Y = 7. Then 600 + 900 + 700 = 2200. Once again, there is no way we can get a *carryover of 4 from the 10's place to make THAT at least 2600.* ***This proves that H cannot be 6****.*

<u>Next, try H = 5</u>. Using the same logic as before, start with the maximum possible values of D and Y: 9 and 8. Then 500 + 900 + 800 = 2200. Since the maximum possible carry from the 10's place is 3, it follows that if H = 5, D and Y ***must*** *be 9 and 8 (in some order); otherwise, THAT will not make it to 2500.*

<u>Begin with D = 9 and Y = 8</u>. We get

```
   5 O W
   9 I 9
   8 O U
+    9 O
 2 5 A 2
```

The remaining digits are 0, 1, 3, 4, 6, and 7. If O is 4 or lower, then the 10's place will never make it to 30, meaning there will not be a carryover of 3 to the 100's place. This means that O must be 6 or 7.

Start with O = 7 (in order to maximize the value of HOW). Then we get

```
   5 7 W
   9 I 9
   8 7 U
+    9 7
 2 5 A 2
```

The digits we have left are 0, 1, 3, 4, and 6. In the 1's place, W + U + 16 must end with 2; the only sum that works is 22. Then W + U + 16 = 22, which means W + U = 6. That means that W and U must be 0 and 6 (in some order).

Then I + 7 + 7 + 9 + 2 (the carryover from the 1's) must make it to 30. This means that I + 25 must be at least 30, so I must be at least 5. The only digits remaining for I are 1, 3, and 4, so this will not work.

Since O cannot be 7, try O = 6. We get

```
   5 6 W
   9 I 9
   8 6 U
+    9 6
 2 5 A 2
```

*Using the same logic as above, the remaining digits are 0, 1, 3, 4, and 7. In the 1's place, W + U + 15 must end with 2. Again, the only sum that works is 22, which means W + U + 15 = 22. Then W + U = 7. W and U can be 0 and 7 *or* 3 and 4.*

Now focus on I. In the 10's place, we get a carryover of 2, so I + 2 + 6 + 6 + 9 must be at least 30. Then I + 23 must be at least 30, so I must be at least 7. This time, 7 is available (!), so plug that in:

```
   5 6 W
   9 7 9
   8 6 U
```

```
+   9 6
-------
2 5 A 2
```

Using I = 7, it follows that A must be exactly 0. 0 is available, so plug that in.

```
  5 6 W
  9 7 9
  8 6 U
+   9 6
-------
 2 5 0 2
```

This leaves 1, 3, and 4 for W and U. Since W and U are interchangeable, pick 4 for W and 3 for U in order to maximize the value of HOW. We get

```
  5 6 4
  9 7 9
  8 6 3
+   9 6
-------
 2 5 0 2
```

At last, we have a solution for HOW that works. But is it the greatest possible HOW?

We still need to <u>try D = 8 and Y = 9</u> (since this could affect the value of HOW). Once again, O must be 6 or 7 (nothing else will result in a carryover of 3 from the 10's place). Start with O = 7 in order to attempt to maximize the value of HOW. We get

```
  5 7 W
  8 I 8
  9 7 U
+   8 7
-------
2 5 A 2
```

The digits available this time are 0, 1, 3, 4, and 6. To get a carry of 3 from the 10's place, I must be 6. We get

```
  5 7 W
  8 6 8
  9 7 U
+   8 7
-------
2 5 A 2
```

Focus on the 1's place. W + U + 15 must end in 2; given the remaining digits, only 22 will work. So W + U + 15 = 22, meaning W + U = 7. This means W and U must be 3 and 4. Since they are interchangeable, choose W = 4 and U = 3 to maximize the value of HOW:

```
  5 7 4
  8 6 8
  9 7 3
+   8 7
-------
2 5 A 2
```

Finally, fill in A = 0, which is still an available digit:

$$\begin{array}{r} \mathbf{5\,7\,4} \\ 8\,6\,8 \\ 9\,7\,3 \\ \underline{+\quad 8\,7} \\ 2\,5\,0\,2 \end{array}$$

It turns out that THAT is 2502, which is the same value as the first time the cryptarithm worked. But this time, HOW is 574 instead of 564. We conclude that the greatest possible value of HOW is ***574****.*

8) **831 Geometry (Area of Circles); Substitution; Probability.**

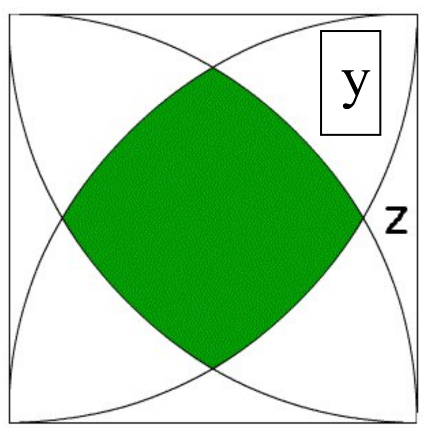
This is a marvelous question.

No, this is a splendiferous question. When Kriti Kaushal presented it to me, I couldn't believe what a wondrous creation it was. Everyone has seen geometry problems, and we've all seen probability problems – but who has ever heard of a probability problem involving geometry?

This was the question of the year for HCMMC, and the solution is as elegant as the question. Let's get started!

<u>Step 1: Calculate the area of each dog's permitted area</u>. The area of a circle is πr^2; for a quarter-circle, it is ¼ of that. Therefore, given a radius of 21 yards and using 22/7 for π, the area of each dog's permitted space is ¼ (22/7) 21^2 = ***693/2****.*

<u>Step 2: Calculate the area of the non-permitted region</u>. In order to solve this problem, we need to figure out a way to "get rid" of the parts that aren't equal to z. Label those four regions y:



It follows that the area of the non-permitted region is equal to the area of the square (21^2) minus the area of the permitted region (693/2). 21^2 = 441, so the area of the non-permitted region is 441 – 693/2 = 189 / 2. Looking at the diagram, we can see that this is equal to 2z + y.

<u>Step 3: Use this new information to determine the area of the overlap</u>. Since 2z + y = 189/2, we can multiply 4 to both sides to get 8z + 4y = 378. What we really want, though, is the value of 4z + 4y, because if we subtract that from the area of the square, we get the area of the overlap. Use algebra:

$$8z + 4y = 378$$
$$4z + 4y = 378 - 4z$$

Since the area of the non-shaded regions is equal to 378 – 4z, it follows that the area of the shaded region (the overlap) is 441 – (378 – 4z) = ***63 + 4z****.*

<u>Step 4: Determine the probability that any one dog is in the overlap area</u>. The overlap (what we want) has an area of 63 + 4z. The permitted area for any dog is 693/2 (what we could get). Then the probability of a dog being inside the overlap is (63 + 4z) / (693 / 2) = 2 (63 + 4z) / 693 = ***(8z + 126) / 693****.*

<u>Step 5: Calculate the probability that all four dogs are in the overlap area simultaneously</u>. Since each dog has an equal probability of being in the overlap area, the overall probability is the number above multiplied to itself four times, or

$$\left[\, (8z + 126) / 693 \,\right]^{4}.$$

<u>Step 6: Find the value of A + B + C + D</u>. A = 8, B = 126, C = 693, and D = 4. Then A + B + C + D = 134 + 697 = ***831****.*

9) **Deep Dish Crust with Ham** **CLT, Logic.** *We like to call this type of pure logic question a "Nelson Special." Kudos to Kavya Kaushal for creating this year's excellent version!*

Begin by creating a CLT of all options. For this chart, toppings are on top, and crusts are on the side. Options that are not available have been shaded.

	Veggies	Pepperoni	Cheese	Ham	Bacon	Pineapple
Thin						
Thick						
Deep Dish						
Gluten Free						

<u>Start with José</u>. Since each crust has more than one type of topping, José doesn't know which pizza is Matty's favorite. However, when he states that he knows that Ricardo also does not know Matty's favorite pizza, he reveals some useful information. If José knew the pizza was thin crust or thick crust, he could <u>not</u> say for certain that Ricardo didn't know which pizza was Matty's favorite, because the thin crust with bacon is the only pizza with bacon, while the thick crust with pepperoni is the only pizza with pepperoni. In those two cases, Ricardo would indeed know Matty's favorite pizza because Ricardo knows the topping, and if the topping were bacon or pepperoni, Ricardo would know Matty's favorite. Since José is certain that Ricardo does not know, we know for sure that Matty's favorite pizza can't be thin crust or thick crust. Eliminate all pizzas in those categories and update the chart:

	Veggies	Pepperoni	Cheese	Ham	Bacon	Pineapple
Thin						
Thick						
Deep Dish						
Gluten Free						

<u>Now focus on Ricardo</u>. Ricardo originally didn't know which pizza Matty liked best, but now he does. This means that the pizza must correspond to a topping that only appears one time in the chart above: veggies, ham, or pineapple. It cannot be cheese because then Ricardo

couldn't say he knew which pizza Matty liked. Update the chart once again:

	Veggies	Pepperoni	Cheese	Ham	Bacon	Pineapple
Thin						
Thick						
Deep Dish						
Gluten Free						

<u>Use the final statement by José</u>. After Ricardo's statement, José also knows which pizza is Matty's favorite. This means that the crust cannot be gluten free, because there are two types of gluten free pizzas still left: if José knew that the crust was gluten free, he still wouldn't know Matty's favorite. But he does know Matty's favorite, so the crust has to be deep dish.

Matty's favorite pizza is therefore ***Deep Dish Crust with Ham****.*

10) **23** **Probability; Cancelling (or Elegant Math Shortcut).** *This is one of those crazy problems that involves a trick – if you can find it – to be solved quickly. But before we get to that point, let's begin by understanding the information.*

<u>Determine the number of Jolly Ranchers</u>. There are 29 + 33 + 12 + 21 + 5 = 100 total candies. Since Mr. G. and Guass pick the same number from the bag, each picks 50. Also, since Mr. G. wishes to ensure a win before Guass eats his first Jolly Rancher, Mr. G. must pick all 50 of his first <u>and</u> eat all of them, so that even if Gauss eats all of his, Mr. G. will win anyway since he ate the first one.

<u>Determine which Jolly Ranchers Mr. G. can eat</u>. Mr. G. will not eat watermelons, nor will he eat blueberries (since they are served with ketchup). This means that he can eat only cherries, grapes, and green apples: he can select from 33 cherries + 12 grapes + 5 green apples = 50 total candies.

<u>Method 1: Determine the probability</u>. Since Mr. G. has 50 candies that work for him out of 100 in the bag, the probability that he will pick a candy that works is 50/100 – on the FIRST try. Since candies are not replaced, there will be 99 candies left in the bag after the first try, of which Mr. G. has 49 that now work. So the probability on the SECOND try is 49/99. On the THIRD try, there are 98 candies left, of which 48 now work: that probability is 48/98. This pattern continues all the way until Mr. G. draws his last candy. There will be 51 candies left in the bag, of which only 1 will work: a probability of 1/51.

To determine the probability of all these events happening, we multiply all the individual probabilities together. We get

$$50/100 \times 49/99 \times 48/98 \times 47/97 \times \ldots \times 4/54 \times 3/53 \times 2/52 \times 1/51.$$

At this point, you can do what I did and use LOTS AND LOTS of cancelling in order to get this down to a fraction in the form of 1/P. You can save some time by circling all the denominators which are prime numbers (those will not reduce since they share no common factors other than 1 with any of the numerators) and by reducing 50 and 100 to 1 and 2, 49 and 98 to 1 and 2, 48 and 96 to 1 and 2, and so on down to 26 and 52. That will eliminate half of the numerators and denominators, making the calculation simpler to perform and easier to see.

The final fraction (after all cancelling) is

$$\frac{1}{2^3 \times 3^4 \times 11 \times 13 \times 17 \times 19 \times 29 \times 31 \times 53 \times 59 \times 61 \times 67 \times 71 \times 73 \times 79 \times 83 \times 89 \times 97}$$

We need the sum of the powers of the denominators. Other than for 2 and 3, the powers are all 1; the sum is thus 3 + 4 + (16 × 1) = 7 + 16 = ***23****.*

This method works, but it takes a long time and it is extremely easy to make a mistake with so many numbers to cancel. Thankfully, I showed this problem to my math geniuses-in-residence, Nelson Niu and Nathaniel Satriya, and both of them independently came up with a far more elegant solution.

<u>Method 2: Factorials; Prime Factors</u>. As you read the solution above, you might have been thinking to yourself, "Isn't there some way this can be calculated without having to write out everything?" Answer: yes, there is a way – but you have to think of the probability in terms of factorials rather than fractions! 50/100 × 49/99 × 48/98 × … × 1/51 can instead be written as

$$\frac{50! \times 50!}{100!}$$

50! represents the numerators from 50 multiplied down to 1. As for the denominators from 100 to 51, we cannot express that as 100! because we are missing 50 down to 1. But if we put another 50! in the numerator, we get 50!/100! which does represent the denominators correctly.

<u>And now for the beautiful solution: go one prime factor at a time</u>. Start with 2's. The number of factors of 2 in 100! can be found by considering how many 2's, 4's, 8's, 16's, 32's, and 64's there are. Simply divide 100 by each of these numbers and ignore the remainders!

2's: 100/2 = 50. 4's: 100/4 = 25. 8's: 100/8 = 12. 16's: 100/16 = 6.
32's: 100/32 = 3. 64's: 100/64 = 1. There are 50 + 25 + 12 + 6 + 3 + 1 = 97 total 2's.

Repeat the process for 50! to get 25 + 12 + 6 + 3 + 1 = 47 total 2's. Then the number of 2's that remain in the denominator after cancelling will be 97 – (47 × 2) = ***3****.*

For 3, it's 100/3 + 100/9 + 100/27 +100/81. Ignoring remainders, we get 33 + 11 + 3 + 1 = 48. Now do this for 50: 50/3 + 50/9 + 50/27 gives us 16 + 5 + 1 = 22. Then the number of 3's remaining in the denominator is 48 – (22 × 2) = ***4****.*

This process gets easier and easier as the prime numbers get larger. Using 5, for example, we only have to check 5^1 and 5^2, since 5^3 is over 100.

<u>5's</u>: (20 + 4) – 2 (10 + 2) = 24 – 24 = 0, so no 5's.
<u>7's</u>: (14 + 2) – 2 (7 + 1) = 16 – 16 = 0, so no 7's.
<u>11's</u>: 9 – 2 (4) = ***1****.*
<u>13's</u>: 7 – 2 (3) = ***1****.*
<u>17's</u>: 5 – 2 (2) = ***1****.*
<u>19's</u>: 5 – 2 (2) = ***1****.*
<u>23's</u>: 4 – 2 (2) = 0.
<u>29's</u>: 3 – 2 (1) = ***1****.*
<u>31's</u>: 3 – 2 (1) = ***1****.*

37's: 2 – 2 (1) = 0.
41's: 2 – 2 (1) = 0.
43's: 2 – 2 (1) = 0.
47's: 2 – 2 (1) = 0.

All the prime numbers that remain in the denominator are not going to cancel, since they share no common factors (other than 1) with any of the numerators. We summarize this by simply listing all prime numbers between 51 and 100:

53, 59, 61, 67, 71, 73, 79, 83, 89, and 97. Each of these will be to the power of 1; the sum of these powers is ***10****.*

To find the total number of powers, add all the numbers. We get 3 + 4 + (1 × 6) + 10 = ***23****. A brilliant solution!*

For those who are curious, here is the actual value of P:

$$\frac{1}{100{,}891{,}344{,}545{,}564{,}193{,}334{,}812{,}497{,}256}$$

11) **3√3 – 11/8** **Geometry; Area of Equilateral Triangles and Circles.**

12) **I'm tired. Solve it on your own.**

Question Composition Credits

Problems	Composer
1, 2, 5, 7, 12	Mr. G.
3, 6, 11	Nathaniel Satriya (TA)
4	Ron Tal
8	Kriti Kaushal (TA)
9	Kavya Kaushal (TA)

10	Jarry Zhou *(with an assist from Mr. G)*