1) **133** **Work Backwards.** *This is a classic WB (Work Backwards) problem: we are given the final result (109) but need to find the original number. To do this, you perform the opposite operation and go one step at a time.*

<u>Add 10</u>: Subtract 10. 109 – 10 = 99.
<u>Multiply by 9</u>: Divide by 9. 99 / 9 = 11.
<u>Subtract 8</u>: Add 8. 11 + 8 = 19.
*<u>Divide by 7</u>: Multiply by 7. 19 x 7 = **133**.*

Confirm by checking. 133 / 7 = 19 – 8 = 11 x 9 = 99 + 10 = 109: YES.

2) **3 ½ (or 3.5)** **Circumference; Conversions.** *84 kilometers = 84,000 meters, so in 1 hour, Grandma Sinko travels 84,000 meters. Also, 400 revolutions per minute means that in 1 hour (or 60 minutes), Samosa makes 400 x 60 = 24,000 revolutions per hour.*

*Since 84,000 meters are covered in 24,000 revolutions, 84 meters are covered in 24 revolutions. 84 / 24 = 7 / 2 = 3 ½ meters per revolution. Since 1 revolution is the same as the circumference of one of Samosa's tires, the answer is **3 ½**.*

3) **53** **Ballparking; Logic.** *The product ends in a 5, so one of the numbers in our list has to be 5 (there are no other prime numbers that end in 5). The only possible sets of five consecutive primes which include 5 are 2, 3, 5, 7, 11 and 3, 5, 7, 11, 13 and 5, 7, 11, 13, 17. Eliminate the first set because the product will be even (85,085 is not even). We are now down to two possibilities.*

Ballpark: 10 x 10 x 10 x 10 x 10 = 100,000, which is pretty close to 85,085. The average of the five numbers in the set should therefore be close to 10. The average of 3, 5, 7, 11, and 13 is 39 / 5, which is less than 8. The average of 5, 7, 11, 13, and 17 is 53 / 5, which is between 10 and 11. The second set is more likely; multiply to see if it works.

*5 x 7 = 35 x 11 = 385 x 13 = 5005 x 17 = 85,085. Yes. The question asks for the sum of these five prime numbers. Add: 5 + 7 + 11 + 13 + 17 = **53**.*

4) **22.** **Fractions; Division**. *Instead of considering how much value is lost in actual dollars, it is much easier to consider what fraction is left over after each investment. Losing 90% is the same as keeping 10%, which is equal to 1/10. Losing 8/9 of that amount means keeping 1/9. It follows that Audrey keeps only 1/10 of 1/9 of 1/8 of 1/7 of 1/6 of 1/5 of 1/4 of 1/3 of 1/2 of the original investment. The amount remaining at the end is equal to the original amount divided by 10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2. We get*

$$\frac{79{,}833{,}600}{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}$$

Start by cancelling 10, 5, and 2 from the bottom. That leaves you with

$$\frac{798{,}336}{9 \times 8 \times 7 \times 6 \times 4 \times 3}$$

*Complete the division. 798,336 / 9 = 88704 / 8 = 11088 / 7 = 1584 / 6 = 264 / 4 = 66 / 3 = **22**.*

5) **–111,111** **Logic**. *To make z as small as possible, w^4, x^3, and y^2 must be as large as possible. Since*

w, x, and y are all less than 20 but greater than 0, assign the highest value to w, the next highest to x, and the next highest to y. This gives w = 19, x = 18, and y = 17. Then $w^4 = 130{,}321$ *and* $x^3 = 5{,}832$ *and* $y^2 = 289$.

This means that 130,321 + 5,832 + 289 + z = 25,331. Solve: 136,442 + z = 25,331. Then z = ***–111,111***.

6) **40** **Logic; Process of Annihilation.** *Originally, I solved this problem by focusing on the middle and right columns, listing all possible digits that could go in those boxes. I determined that the only two digits that didn't work for any of those boxes were 5 and 7; therefore, those digits had to go in the left column. I then figured out which numbers went in which box from there.*

That method works, and it did lead to the correct answer. However, there is a more direct method which is faster and easier. ***Nathaniel*** *provides it as follows:*

This problem has a lot of different restrictions. To make it easier from the start, let's begin solving by using the most restrictive fact: that C × V ÷ Z = 27. We can see from this that C × V has to be a multiple of 27, so C and V have to be either 9 and 3 or 9 and 6.

If they are 9 and 3, Z must be 1. However, since X ÷ Y – Z = 0, there is a problem: if Z is 1, then X ÷ Y must also be 1, which means that X = Y. That is not possible, so ***Z must be 2****, and C and V are 9 and 6.*

Now since T ÷ U – V = 1, V must be 6: if it were 9, then T ÷ U = 10, which is again impossible. Therefore, ***V = 6, meaning that C = 9*** *and T ÷ U = 7. It is then clear that* ***T is 7 and U is 1****, as those are the only digits whose quotient is 7.*

Now, notice that X ÷ Y – Z = 0. Since Z = 2, X ÷ Y must also be 2. Since 2 and 6 have already been used, ***X and Y have to be 8 and 4 respectively****.*

Finally, because B ÷ U × Y = 12, ***B is 3*** *(3 × 4 = 12)* ***and A, the last remaining digit, is 5*** *(5 + 3 + 9 = 17). Our final answer, Y × A × Z, is 4 × 5 × 2, or* ***40****.*

The completed chart looks like this:

5	**3**	**9**
7	**1**	**6**
8	**4**	**2**

7) **1024** **G/C or Logic.** *How in the world are you supposed to solve a problem when it seems like it will have an infinite number of possible solutions?!*

*Answer: **start somewhere** and see if you notice something. We'll present two solutions.*

*Method 1: G/C. This is the solution I used, but I actually prefer Pranay's solution better. Nevertheless, I'm showing both solutions so that you can see there is more than one way to solve such a problem. Consider **31**. If the first block is 31, the remaining cheese blocks cannot be added to 31 to make any new amounts. Similarly, for 30, 29, 28, 27, …, 17: all of these blocks will be too big (in the case of 17, the remaining blocks would need to add up to 14, but that won't cover 15 or 16). So the first block could be **16**, which leaves the remaining 4 blocks to equal 15.*

Try 1 + 1 + 1 + 12: that adds up to 15, but the 1's aren't different sized.
Try 1 + 2 + 3 + 9: this almost works, but there's no way to make 7 or 8.
*Try 1 + 2 + 4 + 8: we can make 1 (1), 2 (2), 3 (1+2), 4 (4), 5 (4+1), 6 (4+2), 7 (1+2+4), and 8 (8). Since we can make all amounts from 1-7, we can make all amounts from 8+1 through 8+7, which covers 9 through 15. Finally, since we can make all amounts from 1 through 15, we can make all amounts from 16 through 31 as well (16, 16+1, 16 +2, and all the way to 16 + 15). The cheese blocks Great Uncle Venkatesh needs are thus 1, 2, 4, 8, and 16. The product of these numbers is 1 × 2 × 4 × 8 × 16 = 64 × 16 = **1024**.*

*Method 2: Logic (solution provided by Pranay Mallik, composer of this problem). The first day of the month is the 1^{st}, so your first nugget has to be **1**. The second day is the 2^{nd}, so you use a **2** (not 1+1 because that repeats a value). The third day is the 3^{rd}, but 1 + 2 = 3, so you don't need a 3. You do need a **4** for the 4^{th} day. Aha! There is a pattern here! We don't need a 5, 6, or 7, but the next one we will need comes right after 1 + 2 + 4 = 7, which is **8**. Then the next one after that will be the one that comes right after 1 + 2 + 4 + 8 = 15, which is **16**.*

*The blocks Great Uncle Venkatesh needs are 1, 2, 4, 8, and 16 (note that 1 + 2 + 4 + 8 + 16 = 31, the final day). The product of these numbers is 1 × 2 × 4 × 8 × 16 = **1024**.*

8) **4** **Casework/Go In Order; Average.** *This is a difficult problem unless you are very careful in the way you try different possibilities. You can focus on the number of quarters (possibly using average to get close), or you can focus on the number of pennies. Note that since the total is 100 cents (and since nickels, dimes, and quarters are always multiples of 5), the number of pennies must be a multiple of 5.*

Try 0 pennies: then 48 coins must make 100. This is not possible, because even if all the coins were as small as possible (nickels), 48 x 5 is way over 100.
Try 5 pennies: then 43 coins must make 95. 43 x 5 is again way over 95; not possible.
Try 10 pennies: then 38 coins must make 90. No.
Try 15 pennies: then 33 coins must make 85. No.
Try 20 pennies: then 28 coins must make 80. No.
Try 25 pennies: then 23 coins must make 75. No.
Try 30 pennies: then 18 coins must make 70. No, but at least that's close (18 x 5 = 90).

*Try 35 pennies: then 13 coins must make 65. **YES**. This can be accomplished with 13 nickels.*

Try 40 pennies: then 8 coins must make 60. Try all different possibilities.

- *With 2 quarters: then 6 coins must make 10. No.*

- *With 1 quarter: then 7 coins must make 35.* ***YES****: this can be accomplished with 7 nickels.*
- *With 0 quarters: then 8 coins must make 60.* ***YES****: this can be accomplished with 4 dimes and 4 nickels.*

Try 45 pennies: then 3 coins must make 55. ***YES****: this can be done with 2 quarters and 1 nickel.*

Thus, 100 cents can be made using 48 coins in ***4*** *different ways (35P+13N, 40P+1Q+7N, 40P+4D+4N, and 45P+2Q+1N).*

9) **5 / 324** **Logic; Casework.** *To determine the probability that Audrey will roll a sum of 7 using 4 6-sided dice, first consider the total number of possible combinations. Audrey could roll any of 6 numbers on each roll, so the total number of combinations that she could roll is 6 x 6 x 6 x 6 (or 6^4) = 36 x 36 = 1296. This is what she could get.*

Next, determine how many ways she could roll a sum of 7 (this is what she wants). If she rolls a 5 or a 6 on any of the dice, the sum will be too high. Begin with a 4.

- *4 + 1 + 1 + 1. The 4 could go in any position (4111, 1411, 1141, or 1114), giving us* ***4*** *ways to make 7 using this combination.*
- *3 + 2 + 1 + 1. There are 4 places we can place the 3, and then 3 places we can place the 2, giving us 4 x 3 =* ***12*** *ways to make 7 using this combination. To illustrate this, start with the 3 and list the combinations: 3211, 3121, 3112, 1321, 1312, 1132. Do the same thing starting with the 2: 2311, 2131, 2113, 1231, 1213, 1123.*
- *2 + 2 + 2 + 1. The 1 could go in any position (2221, 2212, 2122, 1222), giving us* ***4*** *ways to make 7 using this combination.*

Audrey can roll a 7 in 4 + 12 + 4 = 20 ways total. Probability = What You Want / What You Could Get; for this problem, the probability is 20 / 1296. To express this as a fraction in lowest terms, divide both sides by 4 to get ***5/324****.*

10) **459** **Logic; Process of Annihilation (Cryptarithm).** *There are various ways to solve this particular problem. We'll show two methods.*

Method 1: Solve for B. Since BAM + BMA equals a three-digit number, B must be 4 or less, because otherwise the resulting sum would be a four-digit number. Also, B cannot be 0, because then BAM and BMA would not be three-digit numbers. This means B must be 1, 2, 3, or 4.

If B = 1, then M (in the sum MAB) must be 2 or 3. If M is 2, then A must be 9 based on the ones digits (2 + 9 = 11). Then we get 192 + 129 = 291, but that does not work. We will use this same logic as we try the other values for B, M, and A.

If B = 1 and M = 3, then A = 8 and we get 183 + 138 = 381. That doesn't work.

If B = 2, then M must be 4 or 5. Repeat the process: with B = 2 and M =4, we get A = 8. The addition becomes 284 + 248 = 482, which doesn't work. Try B = 2 and M = 5 to obtain A = 7. Then 275 + 257 = 572. Again, it doesn't work.

Let B = 3 so that M must be 6 or 7. If B = 3 and M = 6, then A = 7 and we get 376 + 367 = 673, which doesn't work. If B = 3 and M = 7, then A = 6 and we get 367 + 376 = 763. No.

Let B = 4 so that M must be 8 or 9. If B = 4 and M = 8, then A = 6 and we see that 468 + 486 = 864, which doesn't work. Now try B = 4 and M = 9. Then A = 5 and we get 459 + 495 = 954. This does work! The value of BAM is therefore ***459****.*

<u>Method 2: Consider A and M together</u>. In both the ones place and the tens place, we have A + M. The results are different, though (A and B). Focusing on the ones column, this means that M + A must be equal to B + 10, because there has to be a carryover in order to make the results different. Since M + A gives a carryover from the ones place, it follows that A + M must give a carryover from the tens place.

Putting all this information together, we can see from the tens column that A + M + 1 (the 1 is the carryover from the ones place) must equal A + 10. Create an equation to solve for M:

$$A + M + 1 = A + 10$$
$$M + 1 = 10$$
$$M = 9$$

Now solve for B. The hundreds column has a carryover, so 1 + B + B = 9. Then 2B = 8, and B = 4.

Finally, solve for A. In the ones place, 9 + A = 4. We know there is a carryover, so that becomes 9 + A = 14, giving us A = 5. Check the values: does 459 + 495 = 954? Yes. BAM is ***459****.*

11) **Deep Dish Crust with Ham** **CLT, Logic.** *We like to call this type of pure logic question a "Nelson Special." Kudos to Kavya Kaushal for creating this year's excellent version!*

Begin by creating a CLT of all options. For this chart, toppings are on top, and crusts are on the side. Options that are not available have been shaded.

	Veggies	Pepperoni	Cheese	Ham	Bacon	Pineapple
Thin						
Thick						
Deep Dish						
Gluten Free						

<u>Start with José</u>. Since each crust has more than one type of topping, José doesn't know which pizza is Matty's favorite. However, when he states that he knows that Ricardo also does not know Matty's favorite pizza, he reveals some useful information. If José knew the pizza was thin crust or thick crust, he could <u>not</u> say for certain that Ricardo didn't know which pizza was Matty's favorite, because the thin crust with bacon is the only pizza with bacon, while the thick crust with pepperoni is the only pizza with pepperoni. In those two cases, Ricardo would indeed know Matty's favorite pizza because Ricardo knows the topping, and if the topping were bacon or pepperoni, Ricardo would know Matty's favorite. Since José is certain that Ricardo does not know, we know for sure that Matty's favorite pizza can't be thin crust or thick crust. Eliminate all pizzas in those categories and update the chart:

	Veggies	Pepperoni	Cheese	Ham	Bacon	Pineapple
Thin						
Thick						

Deep Dish						
Gluten Free						

<u>Now focus on Ricardo</u>. Ricardo originally didn't know which pizza Matty liked best, but now he does. This means that the pizza must correspond to a topping that only appears one time in the chart above: veggies, ham, or pineapple. It cannot be cheese because then Ricardo couldn't say he knew which pizza Matty liked. Update the chart once again:

	Veggies	**Pepperoni**	**Cheese**	**Ham**	**Bacon**	**Pineapple**
Thin						
Thick						
Deep Dish						
Gluten Free						

<u>Use the final statement by José</u>. After Ricardo's statement, José also knows which pizza is Matty's favorite. This means that the crust cannot be gluten free, because there are two types of gluten free pizzas still left: if José knew that the crust was gluten free, he still wouldn't know Matty's favorite. But he does know Matty's favorite, so the crust has to be deep dish.

Matty's favorite pizza is therefore ***Deep Dish Crust with Ham****.*

12) **23** **Probability; Cancelling (or Elegant Math Shortcut).** *This is one of those crazy problems that involves a trick – if you can find it – to be solved quickly. But before we get to that point, let's begin by understanding the information.*

<u>Determine the number of Jolly Ranchers</u>. There are 29 + 33 + 12 + 21 + 5 = 100 total candies. Since Mr. G. and Guass pick the same number from the bag, each picks 50. Also, since Mr. G. wishes to ensure a win before Guass eats his first Jolly Rancher, Mr. G. must pick all 50 of his first <u>and</u> eat all of them, so that even if Gauss eats all of his, Mr. G. will win anyway since he ate the first one.

<u>Determine which Jolly Ranchers Mr. G. can eat</u>. Mr. G. will not eat watermelons, nor will he eat blueberries (since they are served with ketchup). This means that he can eat only cherries, grapes, and green apples: he can select from 33 cherries + 12 grapes + 5 green apples = 50 total candies.

<u>Method 1: Determine the probability</u>. Since Mr. G. has 50 candies that work for him out of 100 in the bag, the probability that he will pick a candy that works is 50/100 – on the FIRST try. Since candies are not replaced, there will be 99 candies left in the bag after the first try, of which Mr. G. has 49 that now work. So the probability on the SECOND try is 49/99. On the THIRD try, there are 98 candies left, of which 48 now work: that probability is 48/98. This pattern continues all the way until Mr. G. draws his last candy. There will be 51 candies left in the bag, of which only 1 will work: a probability of 1/51.

To determine the probability of all these events happening, we multiply all the individual probabilities together. We get

$$50/100 \times 49/99 \times 48/98 \times 47/97 \times \ldots \times 4/54 \times 3/53 \times 2/52 \times 1/51.$$

At this point, you can do what I did and use LOTS AND LOTS of cancelling in order to get this down to a fraction in the form of 1/P. You can save some time by circling all the denominators which are prime numbers (those will not reduce since they share no common factors other than 1 with any of the numerators) and by reducing 50 and 100 to 1 and 2, 49 and 98 to 1 and 2, 48 and 96 to 1 and 2, and so on down to 26 and 52. That will eliminate half of the numerators and denominators, making the calculation simpler to perform and easier to see.

The final fraction (after all cancelling) is

$$\frac{1}{2^3 \times 3^4 \times 11 \times 13 \times 17 \times 19 \times 29 \times 31 \times 53 \times 59 \times 61 \times 67 \times 71 \times 73 \times 79 \times 83 \times 89 \times 97}$$

We need the sum of the powers of the denominators. Other than for 2 and 3, the powers are all 1; the sum is thus 3 + 4 + (16 × 1) = 7 + 16 = ***23****.*

This method works, but it takes a long time and it is extremely easy to make a mistake with so many numbers to cancel. Thankfully, I showed this problem to my math geniuses-in-residence, Nelson Niu and Nathaniel Satriya, and both of them independently came up with a far more elegant solution.

<u>Method 2: Factorials; Prime Factors</u>. As you read the solution above, you might have been thinking to yourself, "Isn't there some way this can be calculated without having to write out everything?" Answer: yes, there is a way – but you have to think of the probability in terms of factorials rather than fractions! 50/100 × 49/99 × 48/98 × … × 1/51 can instead be written as

$$\frac{50! \times 50!}{100!}$$

50! represents the numerators from 50 multiplied down to 1. As for the denominators from 100 to 51, we cannot express that as 100! because we are missing 50 down to 1. But if we put another 50! in the numerator, we get 50!/100! which does represent the denominators correctly.

<u>And now for the beautiful solution: go one prime factor at a time</u>. Start with 2's. The number of factors of 2 in 100! can be found by considering how many 2's, 4's, 8's, 16's, 32's, and 64's there are. Simply divide 100 by each of these numbers and ignore the remainders!

2's: 100/2 = 50. 4's: 100/4 = 25. 8's: 100/8 = 12. 16's: 100/16 = 6.
32's: 100/32 = 3. 64's: 100/64 = 1. There are 50 + 25 + 12 + 6 + 3 + 1 = 97 total 2's.

Repeat the process for 50! to get 25 + 12 + 6 + 3 + 1 = 47 total 2's. Then the number of 2's that remain in the denominator after cancelling will be 97 – (47 × 2) = ***3****.*

For 3, it's 100/3 + 100/9 + 100/27 +100/81. Ignoring remainders, we get 33 + 11 + 3 + 1 = 48. Now do this for 50: 50/3 + 50/9 + 50/27 gives us 16 + 5 + 1 = 22. Then the number of 3's remaining in the denominator is 48 – (22 × 2) = ***4****.*

This process gets easier and easier as the prime numbers get larger. Using 5, for example, we only have to check 5^1 and 5^2, since 5^3 is over 100.

<u>5's</u>: (20 + 4) – 2 (10 + 2) = 24 – 24 = 0, so no 5's.
<u>7's</u>: (14 + 2) – 2 (7 + 1) = 16 – 16 = 0, so no 7's.

*<u>11's</u>: 9 – 2 (4) = **1**.*
*<u>13's</u>: 7 – 2 (3) = **1**.*
*<u>17's</u>: 5 – 2 (2) = **1**.*
*<u>19's</u>: 5 – 2 (2) = **1**.*
<u>23's</u>: 4 – 2 (2) = 0.
*<u>29's</u>: 3 – 2 (1) = **1**.*
*<u>31's</u>: 3 – 2 (1) = **1**.*
<u>37's</u>: 2 – 2 (1) = 0.
<u>41's</u>: 2 – 2 (1) = 0.
<u>43's</u>: 2 – 2 (1) = 0.
<u>47's</u>: 2 – 2 (1) = 0.

All the prime numbers that remain in the denominator are not going to cancel, since they share no common factors (other than 1) with any of the numerators. We summarize this by simply listing all prime numbers between 51 and 100:

*<u>53, 59, 61, 67, 71, 73, 79, 83, 89, and 97</u>. Each of these will be to the power of 1; the sum of these powers is **10**.*

*To find the total number of powers, add all the numbers. We get 3 + 4 + (1 × 6) + 10 = **23**. A brilliant solution!*

For those who are curious, here is the actual value of P:

$$\frac{1}{100{,}891{,}344{,}545{,}564{,}193{,}334{,}812{,}497{,}256}$$

Question Composition Credits

Problem	Composer
1	Tejas Rao
2	Sabrina Ning
3-6, 10	Mr. G.
7	Pranay Mallik
8	Ankith Madadi
9	Stephen He
11	Kavya Kaushal
12	Jarry Zhou *(with an assist from Mr. G)*